Global Optimization and Infinity Computing

Marat S. Mukhametzhanov

DIMES, University of Calabria, Italy, and ITMM, Nizhni Novgorod State University, Russia

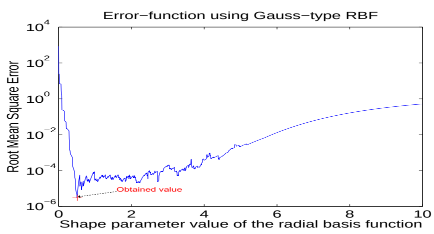
m.mukhametzhanov@dimes.unical.it

Targets of the research

The research consists of two fields: global optimization with expensive and noisy objective functions, and the Infinity Computing – a methodology allowing one to work **numerically** with infinities and infinitesimals. A particular attention is dedicated to application of the Infinity Computer for solving ill-conditioned optimization problems. The research is oriented to solving real-life problems including the following important industrial applications: solution to expensive and ill-conditioned optimization problems in image processing and noisy data fitting; stable and precise solution to ODEs; exact higher order numerical differentiation.

Expensive Global Optimization

Applications in noisy data fitting

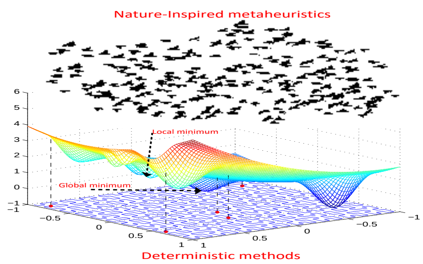


A nonlinear interpolation problem using Radial Basis Functions:

$$RMSE = \sqrt{\frac{1}{s}\sum_{i=1}^{s}|f(x_i) - I_\varepsilon(x_i)|^2} \to \min_{\varepsilon\in\Omega},\ \Omega \subset \mathbb{R}^N,$$

$f(x_i)$, $1 \le i \le s$, – noisy real-valued observations, $R_\varepsilon(r)$ – RBF, $I_\varepsilon(x) = \sum_{k=1}^{m} c_k R_\varepsilon(||x - x_k||_2)$, and ε – the shape parameter. Small $\varepsilon \to$ extremely ill-conditioned systems $\to$ large error.

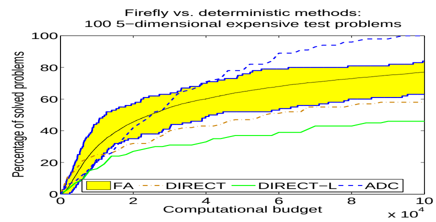
Main optimization framework



A challenging problem: given a limited computational budget, it is required to find a good approximation of the global minimum to a multiparametric and multimodal costly objective function subject to nonlinear constraints.
A promising approach: extension of univariate methods to the multivariable case by means of diagonal space-filling curves ([1, 2]).

Metaheuristic vs Deterministic methods

Metaheuristics (as firefly or other nature-inspired algorithms) are often used to study expensive black-box optimization problems.



The proposed deterministic methods (e.g., based on adaptive diagonal curves, ADC) demonstrate a much better performance with respect to widely used deterministic (e.g., DIRECT) and metaheuristic (e.g., firefly algorithm, FA) methods using the Operational Zones and Aggregated Operational Zones approaches for their comparison (see [1]).

Infinity Computing

Grossone (①)

Grossone – number of elements of the set of natural numbers. The non-contradiction of the methodology has been proven in [3].
The Infinity Computer executes numerically operations with finite, infinite and infinitesimal numbers in a unique framework using ①.
An analogy: amazonian Pirahã tribe can count only **1, 2,** ***many***:

$$many + 1 = many + 2 = many + many = many.$$

Traditional views on infinity:

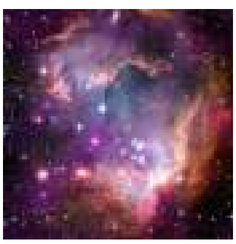
$$\infty + 1 = \infty + 2 = \infty + \infty = \infty$$

Infinity Computing (see [4] and the patents [5] for details):

$$0 \cdot ① = ① - ① = 0^{①} = 0,\ \frac{①}{①} = ①^0 = 1^{①} = 1,\ \frac{①^{1.5}}{①^{2.7}} = ①^{-1.2}.$$

Using ① to stars... and beyond

Today: using ∞



Tomorrow: using ①

Numerical differentiation

Calculate derivatives at the point $x = 3$ of the function f:

$$f(x) = \frac{x+1}{x-1}.$$

The Infinity Computer executes **numerically** the operations

$$f(3 + ①^{-1}) = (3①^0 + ①^{-1} + 1①^0)/(3①^0 + ①^{-1} - 1①^0) =$$
$$= 2①^0 - 0.5①^{-1} + 0.25①^{-2} - 0.125①^{-3} + 0.0625①^{-4} - \ldots$$

From this numeral, we obtain (see [6])

$$f(3) = 2,\ f'(3) = -0.5,\ f''(3) = 2! \cdot 0.25 = 0.5,$$
$$f^{(3)}(3) = 3! \cdot (-0.125) = 0.75,$$

being *exact* values of $f(x)$ and the derivatives at $x = 3$.

Numerical differentiation: ODEs

$$y'(t) = \frac{y - 2ty^2}{1+t},\quad y(0) = y_0 = 0.4,$$

Let us find $y''(0)$ using Euler's method with the step $h = ①^{-1}$ and the forward differences (see [7, 8] for details):

$$y_1 = 0.4 + ①^{-1}f(0, 0.4) = 0.4 + 0.4①^{-1},$$
$$y_2 = y_1 + ①^{-1}f(①^{-1}, y_1) = 0.4 + 0.8①^{-1} - 0.32①^{-2} - 0.32①^{-3},$$
$$y''(0) \approx \frac{\triangle^2_{①^{-1}}}{①^{-2}} = \frac{y_0 - 2y_1 + y_2}{①^{-2}} = \frac{-0.32①^{-2} - 0.32①^{-3}}{①^{-2}} =$$
$$= -0.32 - 0.32①^{-1} = y''(0) + O(①^{-1}).$$

The obtained error is infinitesimal!

Hamiltonian problems

An illustrative application: Fermi-Pasta-Ulam problem (FPU problem, see [9] for more applications, e.g., Kepler problem).

$$H(q,p) = \frac{1}{2}\sum_{i=1}^{m}(p_{2i-1}^2 + p_{2i}^2) + \frac{\omega^2}{4}\sum_{i=1}^{m}(q_{2i} - q_{2i-1})^2 + \sum_{i=0}^{m}(q_{2i+1} - q_{2i})^4$$

$q_0 = q_{2m+1} = 0$, $p_i = \dot{q}_i$, $i = 1, \ldots, 2m$, and $\omega = const$. The total energy $I = I_1 + \ldots + I_m$ of the linear springs is almost conserved, where $I_j = \frac{1}{4}\left((p_{2i} - p_{2i-1})^2 + \omega^2(q_{2i} + q_{2i-1})\right)$.

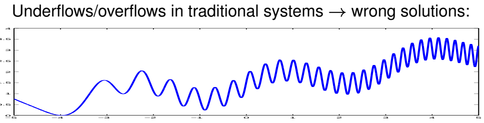
FPU problem: energy

Solved by the Euler-Maclaurin methods using ① with $m = 3$.

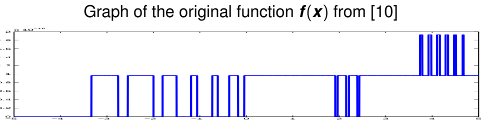
Infinity Computing in Optimization

Traditional computers: ill-conditioning

Underflows/overflows in traditional systems $\to$ wrong solutions:



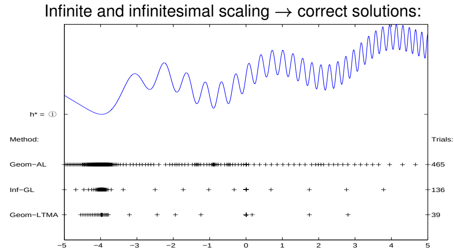
Graph of the original function $f(x)$ from [10]



Graph of the scaled function $g(x) = 10^{-16}f(x) + 1$.

Infinity Computer: well-conditioning

Infinite and infinitesimal scaling $\to$ correct solutions:



Results of three proposed algorithms on the scaled function $g(x) = ①^{-1}f(x) + ①$ from [10, 11].

Constrained optimization: exact penalty

$$\min_x \frac{1}{2}x_1^2 + \frac{1}{6}x_2^2$$
subject to $x_1 + x_2 = 1$

Penalty approach:

$$\min_x \frac{1}{2}x_1^2 + \frac{1}{6}x_2^2 + \frac{P}{2}(1 - x_1 - x_2)^2.$$

Traditional computers – iterative procedures with different P can return approximated solutions only.
Infinity Computer – exact penalty $P = ①$ (see [12]):

$$x_1^* = \frac{1}{4} - ①^{-1}\left(\frac{1}{16} - \frac{1}{64}①^{-1} + \ldots\right),\ x_2^* = \frac{3}{4} - ①^{-1}\left(\frac{3}{16} - \frac{3}{64}①^{-1} + \ldots\right)$$

The finite parts of x_1^* and x_2^* give us the exact solution to the original constrained problem: $\bar{x} = \left(\frac{1}{4}, \frac{3}{4}\right)$

Obtained results

New powerful multivariable optimization schemes have been proposed: global optimization algorithms based on adaptive diagonal curves ([1, 2, 15, 16]), acceleration techniques in derivative-free and smooth global optimization ([11, 13]), ①-based penalty functions in constrained optimization ([12]), where a new generator of test problems with non-linear constraints based on the GKLS-generator for testing algorithms of constrained global optimization has been introduced. New simple and powerful higher order numerical methods for solving ordinary differential equations have been proposed using the Infinity Computing framework ([7–9]). The Infinity Computer has been applied to handling ill-conditioning in optimization ([10, 14]). Presented techniques can be used in different fields, where ill-conditioning appears.

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Università della Calabria – Dipartimento di Ingegneria Informatica, Modellistica, Elettronica e Sistemistica DIMES
Lobachevsky University
RSF Russian Science Foundation
iSoGO
INdAM Istituto Nazionale di Alta Matematica Francesco Severi